ITERATED HARDY-TYPE INEQUALITIES INVOLVING SUPREMA

AMIRAN GOGATISHVILI AND RZA MUSTAFAYEV

ABSTRACT. In this paper the complete solution of the restricted inequalities for supremal operators are given. The boundedness of the composition of supremal operators with the Hardy and Copson operators in weighted Lebesgue spaces are characterized.

1. INTRODUCTION

Throughout the paper we assume that $I := (a, b) \subseteq (0, \infty)$. By $\mathfrak{M}(I)$ we denote the set of all measurable functions on I. The symbol $\mathfrak{M}^+(I)$ stands for the collection of all $f \in \mathfrak{M}(I)$ which are non-negative on I, while $\mathfrak{M}^+(I; \downarrow)$ and $\mathfrak{M}^+(I; \uparrow)$ are used to denote the subset of those functions which are non-increasing and non-decreasing on I, respectively. When $I = (0, \infty)$, we write simply $\mathfrak{M}^\downarrow$ and $\mathfrak{M}^\uparrow$ instead of $\mathfrak{M}^+(I; \downarrow)$ and $\mathfrak{M}^+(I; \uparrow)$, accordingly. The family of all weight functions (also called just weights) on I, that is, locally integrable non-negative functions on $(0, \infty)$, is given by $\mathcal{W}(I)$.

For $p \in (0, \infty]$ and $w \in \mathfrak{M}^+(I)$, we define the functional $\| \cdot \|_{p,w,I}$ on $\mathfrak{M}(I)$ by

$$\|f\|_{p,w,I} := \begin{cases} \left(\int_I |f(x)|^p w(x) \, dx \right)^{1/p} & \text{if} \quad p < \infty \\ \text{ess sup}_I |f(x)| w(x) & \text{if} \quad p = \infty. \end{cases}$$

If, in addition, $w \in \mathcal{W}(I)$, then the weighted Lebesgue space $L^p(w, I)$ is given by

$$L^p(w, I) = \{ f \in \mathfrak{M}(I) : \|f\|_{p,w,I} < \infty \}$$

and it is equipped with the quasi-norm $\| \cdot \|_{p,w,I}$.

When $w \equiv 1$ on I, we write simply $L^p(I)$ and $\| \cdot \|_{p,I}$ instead of $L^p(w, I)$ and $\| \cdot \|_{p,w,I}$, respectively.

Given a operator $T : \mathfrak{M}^+ \to \mathfrak{M}^+$, for $0 < p < \infty$ and $u \in \mathfrak{M}^+$, denote by

$$T_u(g) := T(gu), \qquad g \in \mathfrak{M}^+.$$

Suppose f be a measurable a.e. finite function on $\mathbb{R}^n$. Then its non-increasing rearrangement f^* is given by

$$f^*(t) = \inf\{\lambda > 0 : |\{x \in \mathbb{R}^n : |f(x)| > \lambda\}| \leq t\}, \quad t \in (0, \infty),$$

and let f^{**} denotes the Hardy-Littlewood maximal function of f, i.e.

$$f^{**}(t) := \frac{1}{t} \int_0^t f^*(\tau) \, d\tau, \quad t > 0.$$

Quite many familiar function spaces can be defined using the non-increasing rearrangement of a function. One of the most important classes of such spaces are the so-called classical Lorentz spaces.

Let $p \in (0, \infty)$ and $w \in \mathcal{W}$. Then the classical Lorentz spaces $\Lambda^p(w)$ and $\Gamma^p(w)$ consist of all functions $f \in \mathfrak{M}$ for which $\|f\|_{\Lambda^p(w)} < \infty$ and $\|f\|_{\Gamma^p(w)} < \infty$, respectively. Here it is

$$\|f\|_{\Lambda^p(w)} := \|f^*\|_{p,w,(0,\infty)} \qquad \text{and} \qquad \|f\|_{\Gamma^p(w)} := \|f^{**}\|_{p,w,(0,\infty)}.$$

For more information about the Lorentz Λ and Γ see e.g. [4] and the references therein.

The Hardy and Copson operators are defined by

$$Hg(t) := \int_0^t g(s)\,ds, \qquad g \in \mathfrak{M}^+,$$

and

$$H^*g(t) := \int_t^\infty g(s)\,ds, \qquad g \in \mathfrak{M}^+,$$

respectively. The operators H and H^* play a prominent role. There are other operators that are also of interest. For example, certain specific problems such as the description of the behaviour of the fractional maximal operator on classical Lorentz spaces [6] or the optimal pairing problem for Sobolev imbeddings [17] or various questions arising in the interpolation theory can be handles in an elegant way with the help of the supremal operators

$$S g(t) := \operatorname*{ess\,sup}_{0 < \tau \le t} g(\tau), \qquad g \in \mathfrak{M}^+,$$

and

$$S^*g(t) := \operatorname*{ess\,sup}_{t \le \tau < \infty} g(\tau), \qquad g \in \mathfrak{M}^+.$$

In this paper, we give complete characterization of restricted inequalities

$$(1.1) \qquad \|S_u(f)\|_{q,w,(0,\infty)} \le c\|f\|_{p,v,(0,\infty)}, \ f \in \mathfrak{M}^\downarrow,$$

$$(1.2) \qquad \|S_u(f)\|_{q,w,(0,\infty)} \le c\|f\|_{p,v,(0,\infty)}, \ f \in \mathfrak{M}^\uparrow$$

and

$$(1.3) \qquad \|S_u^*(f)\|_{q,w,(0,\infty)} \le c\|f\|_{p,v,(0,\infty)}, \ f \in \mathfrak{M}^\uparrow,$$

$$(1.4) \qquad \|S_u^*(f)\|_{q,w,(0,\infty)} \le c\|f\|_{p,v,(0,\infty)}, \ f \in \mathfrak{M}^\downarrow.$$

Note that inequality (1.1) was characterized in [14]. It should be mentioned here that it was done under some additional condition on weight function u, when $q < p$ (cf. [14, Theorem 3.4]).

In particular, we characterize the validity of the iterated Hardy-type inequalities involving suprema

$$(1.5) \qquad \left\| S_u \left(\int_0^x h \right) \right\|_{q,w,(0,\infty)} \le c \, \|h\|_{p,v,(0,\infty)},$$

and

$$(1.6) \qquad \left\| S_u \left(\int_x^\infty h \right) \right\|_{q,w,(0,\infty)} \le c \, \|h\|_{p,v,(0,\infty)},$$

where $0 < q < \infty$, $1 \le p < \infty$, u, w and v are weight functions on $(0, \infty)$.

It is worth to mentoin that the characterizations of "dual" inequalities

$$(1.7) \qquad \left\| S_u^* \left(\int_x^\infty h \right) \right\|_{q,w,(0,\infty)} \le c \, \|h\|_{p,v,(0,\infty)},$$

and

$$(1.8) \qquad \left\| S_u^* \left(\int_0^x h \right) \right\|_{q,w,(0,\infty)} \le c \, \|h\|_{p,v,(0,\infty)},$$

can be easily obtained from the solutions of inequalities (1.5) - (1.6), respectively, by change of variables. Note that inequality (1.8) has been characterized in [14] in the case $0 < q < \infty$, $1 \le p < \infty$.

We pronounce that the characterizations of inequalities (1.5) - (1.8) are important because many inequalities for classical operators can be reduced to them (for illustrations of this important fact, see, for instance, [12]). These inequalities play an important role in the theory of Morrey spaces and other topics (see [1], [2] and [3]).

The fractional maximal operator, M_γ, $\gamma \in (0, n)$, is defined at $f \in L^1_{loc}(\mathbb{R}^n)$ by

$$(M_\gamma f)(x) = \sup_{Q \ni x} |Q|^{\gamma/n-1} \int_Q |f(y)| \, dy, \quad x \in \mathbb{R}^n,$$

where the supremum is extended over all cubes $Q \subset \mathbb{R}^n$ with sides parallel to the coordinate axes. It was shown in [6, Theorem 1.1] that

$$(1.9) \qquad (M_\gamma f)^*(t) \lesssim \sup_{t \leq \tau < \infty} \tau^{\gamma/n-1} \int_0^\tau f^*(y) \, dy \lesssim (M_\gamma \tilde{f})^*(t),$$

for every $f \in L^1_{loc}(\mathbb{R}^n)$ and $t \in (0, \infty)$, where $\tilde{f}(x) = f^*(\omega_n |x|^n)$ and ω_n is the volume of S^{n-1}. Thus, in order to characterize boundedness of the fractional maximal operator M_γ between classical Lorentz spaces it is necessary and sufficient to characterize the validity of the weighted inequality

$$\left(\int_0^\infty \left[\sup_{t \leq \tau < \infty} \tau^{\gamma/n-1} \int_0^\tau \varphi(y) \, dy \right]^q w(t) \, dt \right)^{1/q} \lesssim \left(\int_0^\infty [\varphi(t)]^p v(t) \, dt \right)^{1/p}$$

for all $\varphi \in \mathfrak{M}^\downarrow$. This last estimate can be interpreted as a restricted weighted inequality for the operator T_γ, defined by

$$(1.10) \qquad (T_\gamma g)(t) = \sup_{t \leq \tau < \infty} \tau^{\gamma/n-1} \int_0^\tau g(y) \, dy, \quad g \in \mathfrak{M}^+(0, \infty), \quad t \in (0, \infty).$$

Such a characterization was obtained in [6] for the particular case when $1 < p \leq q < \infty$ and in [18, Theorem 2.10] in the case of more general operators and for extended range of p and q. Full proofs and some further extensions and applications can be found in [9], [10].

The operator T_γ is a typical example of what is called a Hardy-operator involving suprema

$$(T_u g)(t) := \sup_{t \leq s < \infty} \frac{u(s)}{s} \int_0^s g(y) \, dy,$$

which combines both the operations (integration and taking the supremum).

In the above-mentioned applications, it is required to characterize a restricted weighted inequality for T_u. This amounts to finding a necessary and sufficient condition on a triple of weights (u, v, w) such that the inequality

$$(1.11) \qquad \left(\int_0^\infty \left(\sup_{t \leq s < \infty} u(s) f^{**}(s) \right)^q w(t) \, dt \right)^{1/q} \lesssim \left(\int_0^\infty f^*(t)^p v(t) \, dt \right)^{1/p}$$

holds. Particular examples of such inequalities were studied in [6] and, in a more systematic way, in [14]. Inequality (1.11) was investigated in [15] in the case when $0 < p \leq 1$. The approach used in this paper was based on a new type reduction theorem which showed connection between three types of restricted weighted inequalities.

Rather interestingly, such operators have been recently encountered in various research projects. They have been found indispensable in the search for optimal pairs of rearrangement-invariant norms for wich a Sobolev-type inequality holds (cf. [17]). They constitute a very useful tool for characterization of the assocaiate norm of an operator-induced norm, which naturally appears as an optimal domain norm in a Sobolev embedding (cf. [19], [20]). Supremum operators are also very useful in limitimg interpolation theory as can be seen from their appearance for example in [11], [8], [7], [21].

Definition 1.1. Let $u \in \mathcal{W}(0, \infty) \cap C(0, \infty)$, $b \in \mathcal{W}(0, \infty)$ and $B(t) := \int_0^t b(s) \, ds$. Assume that b be such that $0 < B(t) < \infty$ for every $t \in (0, \infty)$. The operator $T_{u,b}$ is defined at $g \in \mathfrak{M}^+(0, \infty)$ by

$$(T_{u,b} g)(t) := \sup_{t \leq \tau < \infty} \frac{u(\tau)}{B(\tau)} \int_0^\tau g(s) b(s) \, ds, \qquad t \in (0, \infty).$$

The operator T_γ, defined in (1.10), is a particular example of operators $T_{u,b}$. These operators are investigated in [14] and [15].

In this paper we give complete characterization for the inequality

(1.12) $\|T_{u,b}f\|_{q,w,(0,\infty)} \leq c\|f\|_{p,v,(0,\infty)}, \qquad f \in \mathfrak{M}^\downarrow(0, \infty)$

for $0 < q \leq \infty$, $0 < p < \infty$ (see Theorems 5.1 and 5.5).

Inequality (1.12) was characterized in [14, Theorem 3.5] under additional condition

$$\sup_{0<t<\infty} \frac{u(t)}{B(t)} \int_0^t \frac{b(\tau)}{u(\tau)} \, d\tau < \infty.$$

Note that the case when $0 < p \leq 1 < q < \infty$ was not considered in [14]. It is also worse to mention that in the case when $1 < p < \infty$, $0 < q < p < \infty$, $q \neq 1$ [14, Theorem 3.5] contains only discrete condition. In [15] the new reduction theorem was obtained when $0 < p \leq 1$, and this technique allowed to characterize inequality (1.12) when $b \equiv 1$, and in the case when $0 < q < p \leq 1$ this paper contains only discrete condition.

The paper is organized as follows. Section 2 contains some preliminaries along with the standard ingredients used in the proofs. Full characterization of inequalities (1.1) - (1.4) and (1.5) - (1.7) are given in Sections 3 and 4. Finally, solution of inequality (1.12) are obtained in Section 5.

2. Notations and Preliminaries

Throughout the paper, we always denote by c or C a positive constant, which is independent of main parameters but it may vary from line to line. However a constant with subscript such as c_1 does not change in different occurrences. By $a \lesssim b$, $(b \gtrsim a)$ we mean that $a \leq \lambda b$, where $\lambda > 0$ depends on inessential parameters. If $a \lesssim b$ and $b \lesssim a$, we write $a \approx b$ and say that a and b are equivalent. We will denote by $\mathbf{1}$ the function $\mathbf{1}(x) = 1$, $x \in (0, \infty)$. Unless a special remark is made, the differential element dx is omitted when the integrals under consideration are the Lebesgue integrals. Everywhere in the paper, u, v and w are weights.

We need the following notations:

$$V(t) := \int_0^t v, \quad V_*(t) := \int_t^\infty v,$$

$$W(t) := \int_0^t w, \quad W_*(t) := \int_t^\infty w.$$

Convention 2.1. We adopt the following conventions:

(i) Throughout the paper we put $0 \cdot \infty = 0$, $\infty/\infty = 0$ and $0/0 = 0$.

(ii) If $p \in [1, +\infty]$, we define p' by $1/p + 1/p' = 1$.

(iii) If $0 < q < p < \infty$, we define r by $1/r = 1/q - 1/p$.

(iv) If $I = (a, b) \subseteq \mathbb{R}$ and g is monotone function on I, then by $g(a)$ and $g(b)$ we mean the limits $\lim_{x\to a+} g(x)$ and $\lim_{x\to b-} g(x)$, respectively.

We recall some reduction theorems for positive monotone operators from [16] and [13]. The following conditions will be used below:

(i) $T(\lambda f) = \lambda T f$ for all $\lambda \geq 0$ and $f \in \mathfrak{M}^+$;

(ii) $T f(x) \leq c T g(x)$ for almost all $x \in \mathbb{R}_+$ if $f(x) \leq g(x)$ for almost all $x \in \mathbb{R}_+$, with constant $c > 0$ independent of f and g;

(iii) $T(f + \lambda\mathbf{1}) \leq c(T f + \lambda T\mathbf{1})$ for all $f \in \mathfrak{M}^+$ and $\lambda \geq 0$, with a constant $c > 0$ independent of f and λ.

Theorem 2.2 ([16], Theorem 3.1). *Let $0 < q \leq \infty$ and $1 \leq p < \infty$, and let $T : \mathfrak{M}^+ \to \mathfrak{M}^+$ be an operator. Then the inequality*

(2.1) $\|T f\|_{q,w,(0,\infty)} \leq c\|f\|_{p,v,(0,\infty)}, \qquad f \in \mathfrak{M}^\downarrow(0, \infty)$

implies the inequality

(2.2) $\left\|T\left(\int_x^\infty h\right)\right\|_{q,w,(0,\infty)} \leq c\|h\|_{p,V^p v^{1-p},(0,\infty)}, \qquad h \in \mathfrak{M}^+(0, \infty).$

If $V(\infty) = \infty$ and if T is an operator satisfying conditions (i)-(ii), *then the condition* (2.2) *is sufficient for inequality* (2.1) *to hold on the cone $\mathfrak{M}^{\downarrow}$. Further, if $0 < V(\infty) < \infty$, then a sufficient condition for* (2.1) *to hold on $\mathfrak{M}^{\downarrow}$ is that both* (2.2) *and*

$$\|T\mathbf{1}\|_{q,w,(0,\infty)} \leq c\|\mathbf{1}\|_{p,v,(0,\infty)} \tag{2.3}$$

hold in the case when T satisfies the conditions (i)-(iii).

Theorem 2.3 ([16], Theorem 3.2). *Let $0 < q \leq \infty$ and $1 \leq p < \infty$, and let $T : \mathfrak{M}^+ \to \mathfrak{M}^+$ satisfies conditions* (i) *and* (ii). *Then a sufficient condition for inequality* (2.1) *to hold is that*

$$\left\|T\left(\frac{1}{V^2(x)}\int_0^x hV\right)\right\|_{q,w,(0,\infty)} \leq c\|h\|_{p,v^{1-p},(0,\infty)}, \qquad h \in \mathfrak{M}^+(0,\infty). \tag{2.4}$$

Moreover, (2.1) *is necessary for* (2.4) *to hold if conditions* (i)-(iii) *are all satisfied.*

Theorem 2.4 ([16], Theorem 3.3). *Let $0 < q \leq \infty$ and $1 \leq p < \infty$, and let $T : \mathfrak{M}^+ \to \mathfrak{M}^+$ be an operator. Then the inequality*

$$\|Tf\|_{q,w,(0,\infty)} \leq c\|f\|_{p,v,(0,\infty)}, \qquad f \in \mathfrak{M}^{\uparrow}(0,\infty) \tag{2.5}$$

implies the inequality

$$\left\|T\left(\int_0^x h\right)\right\|_{q,w,(0,\infty)} \leq c\|h\|_{p,V_*^p v^{1-p},(0,\infty)}, \qquad h \in \mathfrak{M}^+(0,\infty). \tag{2.6}$$

If $V_*(0) = \infty$ and if T is an operator satisfying the conditions (i)-(ii), *then the condition* (2.6) *is sufficient for inequality* (2.5) *to hold. If $0 < V_*(0) < \infty$ and T is an operator satisfying the conditions* (i)-(iii), *then* (2.5) *follows from* (2.6) *and* (2.3).

Theorem 2.5 ([16], Theorem 3.4). *Let $0 < q \leq \infty$ and $1 \leq p < \infty$, and let $T : \mathfrak{M}^+ \to \mathfrak{M}^+$ satisfies conditions* (i) *and* (ii). *Then a sufficient condition for inequality* (2.5) *to hold is that*

$$\left\|T\left(\frac{1}{V_*^2(x)}\int_x^\infty hV_*\right)\right\|_{q,w,(0,\infty)} \leq c\|h\|_{p,v^{1-p},(0,\infty)}, \qquad h \in \mathfrak{M}^+(0,\infty). \tag{2.7}$$

Moreover, (2.5) *is necessary for* (2.7) *to hold if conditions* (i)-(iii) *are all satisfied.*

Theorem 2.6. [13, Theorem 3.1] *Let $0 < q \leq \infty$, $1 < p < \infty$, and let $T : \mathfrak{M}^+ \to \mathfrak{M}^+$ satisfies conditions* (i)-(iii). *Assume that u, $w \in \mathcal{W}(0,\infty)$ and $v \in \mathcal{W}(0,\infty)$ be such that*

$$\int_0^x v^{1-p'}(t)\,dt < \infty \qquad \text{for all} \qquad x > 0. \tag{2.8}$$

Then inequality

$$\left\|T\left(\int_0^x h\right)\right\|_{q,w,(0,\infty)} \leq c\,\|h\|_{p,v,(0,\infty)}, \; h \in \mathfrak{M}^+, \tag{2.9}$$

holds iff

$$\|T_{\Phi^2}f\|_{q,w,(0,\infty)} \leq c\|f\|_{p,\phi,(0,\infty)}, \; f \in \mathfrak{M}^{\downarrow}, \tag{2.10}$$

holds, where

$$\phi(x) \equiv \phi[v;p](x) := \left(\int_0^x v^{1-p'}(t)\,dt\right)^{-p'/(p'+1)} v^{1-p'}(x)$$

and

$$\Phi(x) \equiv \Phi[v;p](x) = \int_0^x \phi(t)\,dt = \left(\int_0^x v^{1-p'}(t)\,dt\right)^{1/(p'+1)}.$$

Theorem 2.7. [13, Theorem 3.11] *Let $0 < q \le \infty$, and let $T : \mathfrak{M}^+ \to \mathfrak{M}^+$ satisfies conditions* (i)-(iii). *Assume that u, $w \in \mathcal{W}(0, \infty)$ and $v \in \mathcal{W}(0, \infty)$ be such that $V(x) < \infty$ for all $x > 0$. Then inequality*

$$(2.11) \qquad \left\| T\left(\int_0^x h \right) \right\|_{q,w,(0,\infty)} \le c \, \|h\|_{1,V^{-1},(0,\infty)}, \ h \in \mathfrak{M}^+,$$

holds iff

$$(2.12) \qquad \|T_{V^2} f\|_{q,w,(0,\infty)} \le c \|f\|_{1,v,(0,\infty)}, \ f \in \mathfrak{M}^\downarrow.$$

3. SUPREMAL OPERATORS ON THE CONE OF MONOTONE FUNCTIONS

In this section, we give complete characterization of inequalities (1.1) - (1.4).
To state the next statements we need the following notations:

$$\overline{u}(t) := \sup_{0 < \tau \le t} u(\tau), \qquad \underline{u}(t) := \sup_{t \le \tau < \infty} u(\tau), \qquad (t > 0).$$

For a given weight v, $0 \le a < b \le \infty$ and $1 \le p < \infty$, we denote

$$\sigma_p(a, b) = \begin{cases} \left(\int_a^b [v(t)]^{1-p'} dt \right)^{1/p'} & \text{when } 1 < p < \infty \\ \operatorname*{ess\,sup}_{a<t<b} [v(t)]^{-1} & \text{when } p = 1. \end{cases}$$

Recall the following theorem.

Theorem 3.1. [14, Theorems 4.1 and 4.4] *Let $1 \le p < \infty$, $0 < q < \infty$ and let $u \in \mathcal{W}(0, \infty) \cap C(0, \infty)$. Assume that v, $w \in \mathcal{W}(0, \infty)$ be such that*

$$0 < V(x) < \infty \qquad and \qquad 0 < W(x) < \infty \qquad for \ all \qquad x > 0.$$

Then inequality (1.8) *is satisfied with the best constant c if and only if:*
(i) *$p \le q$, and in this case $c \approx A_1$, where*

$$A_1 := \sup_{x>0} \left([\underline{u}]^q(x) W(x) + \int_x^\infty [\underline{u}]^q(t) w(t) \, dt \right)^{1/q} \sigma_p(0, x);$$

(ii) *$q < p$, and in this case $c \approx B_1 + B_2$, where*

$$B_1 := \left(\int_0^\infty \left(\int_x^\infty [\underline{u}]^q(t) w(t) \, dt \right)^{r/p} [\underline{u}]^q(x) \Big[\sigma_p(0, x) \Big]^r w(x) \, dx \right)^{1/r},$$

$$B_2 := \left(\int_0^\infty W^{r/p}(x) \Big[\sup_{x \le \tau < \infty} \underline{u}(\tau) \sigma_p(0, \tau) \Big]^r w(x) \, dx \right)^{1/r}.$$

Using change of variables $x = 1/t$, we can easily obtain the following statement.

Theorem 3.2. *Let $1 \le p < \infty$, $0 < q < \infty$ and let $u \in \mathcal{W}(0, \infty) \cap C(0, \infty)$. Assume that v, $w \in \mathcal{W}(0, \infty)$ be such that*

$$0 < V_*(x) < \infty \qquad and \qquad 0 < W_*(x) < \infty \qquad for \ all \qquad x > 0.$$

Then inequality (1.6) *is satisfied with the best constant c if and only if:*
(i) *$p \le q$, and in this case $c \approx A_1$, where*

$$A_1 := \sup_{x>0} \left([\overline{u}]^q(x) W_*(x) + \int_0^x [\overline{u}]^q(t) w(t) \, dt \right)^{1/q} \sigma_p(x, \infty);$$

(ii) *$q < p$, and in this case $c \approx B_1 + B_2$, where*

$$B_1 := \left(\int_0^\infty \left(\int_0^x [\overline{u}]^q(t) w(t) \, dt \right)^{r/p} [\overline{u}]^q(x) \Big[\sigma_p(x, \infty) \Big]^r w(x) \, dx \right)^{1/r},$$

$$B_2 := \left(\int_0^\infty W_*^{r/p}(x) \Big[\sup_{0 < \tau \le x} \overline{u}(\tau) \sigma_p(\tau, \infty) \Big]^r w(x) \, dx \right)^{1/r}.$$

Proof. Obviously, inequality (1.6) is satisfied with the best constant c if and only if

$$(3.1) \qquad \left\| S_{\tilde{u}}^* \left(\int_0^x h \right) \right\|_{q,\tilde{w},(0,\infty)} \le c \, \|h\|_{p,\tilde{v},(0,\infty)}, \ h \in \mathfrak{M}^+$$

holds, where

$$\tilde{u}(t) = u\left(\frac{1}{t}\right), \ \tilde{w}(t) = w\left(\frac{1}{t}\right)\frac{1}{t^2}, \ \tilde{v}(t) = v\left(\frac{1}{t}\right)\left(\frac{1}{t^2}\right)^{1-p}, \ t > 0.$$

Using Theorem 3.1, and then applying substitution of variables mentioned above three times, we get the statement. $\qquad \square$

Theorem 3.3. *Let $0 < p, q < \infty$ and let $u \in \mathcal{W}(0,\infty) \cap C(0,\infty)$. Assume that $v, w \in \mathcal{W}(0,\infty)$ be such that $0 < V_*(x) < \infty$ and $0 < W_*(x) < \infty$ for all $x > 0$. Then inequality (1.1) is satisfied with the best constant c if and only if:*

(i) *$p \le q$, and in this case $c \approx A_1 + \|S_u(\mathbf{1})\|_{q,w,(0,\infty)}/\|\mathbf{1}\|_{s,v,(0,\infty)}$, where*

$$A_1 := \sup_{x>0} \left([\overline{u}]^q(x) W_*(x) + \int_0^x [\overline{u}]^q(t) w(t)\, dt \right)^{1/q} V^{-1/p}(x);$$

(ii) *$q < p$, and in this case $c \approx B_1 + B_2 + \|S_u(\mathbf{1})\|_{q,w,(0,\infty)}/\|\mathbf{1}\|_{s,v,(0,\infty)}$, where*

$$B_1 := \left(\int_0^\infty \left(\int_0^x [\overline{u}]^q(t) w(t)\, dt \right)^{r/p} [\overline{u}]^q(x) V^{-r/p}(x) w(x)\, dx \right)^{1/r},$$

$$B_2 := \left(\int_0^\infty W_*^{r/p}(x) \left[\sup_{0<\tau\le x} \overline{u}(\tau) V^{-1/p}(\tau) \right]^r w(x)\, dx \right)^{1/r}.$$

Proof. It is easy to see that inequality (1.1) holds if and only if

$$(3.2) \qquad \|S_{u^p}(f)\|_{q/p,w,(0,\infty)} \le c^p \|f\|_{1,v,(0,\infty)}, \ f \in \mathfrak{M}^{\downarrow}$$

holds. By Theorem 2.2, (3.2) holds iff both

$$(3.3) \qquad \left\| S_{u^p} \left(\int_x^\infty h \right) \right\|_{q/p,w,(0,\infty)} \le c^p \|h\|_{1,V,(0,\infty)}, \ h \in \mathfrak{M}^+,$$

and

$$(3.4) \qquad \|S_u(\mathbf{1})\|_{q,w,(0,\infty)} \le c \|\mathbf{1}\|_{s,v,(0,\infty)}$$

hold. In order to complete the proof, it remains to apply Theorem 3.2. $\qquad \square$

Using change of variables $x = 1/t$, we can easily obtain the following "dual" statement.

Theorem 3.4. *Let $0 < p, q < \infty$ and let $u \in \mathcal{W}(0,\infty) \cap C(0,\infty)$. Assume that $v, w \in \mathcal{W}(0,\infty)$ be such that $0 < V(x) < \infty$ and $0 < W(x) < \infty$ for all $x > 0$. Then (1.3) is satisfied with the best constant c if and only if:*

(i) *$p \le q$, and in this case $c \approx A_1 + \|S_u^*(\mathbf{1})\|_{q,w,(0,\infty)}/\|\mathbf{1}\|_{s,v,(0,\infty)}$, where*

$$A_1 := \sup_{x>0} \left([\underline{u}]^q(x) W(x) + \int_x^\infty [\underline{u}]^q(t) w(t)\, dt \right)^{1/q} V_*^{-1/p}(x);$$

(ii) *$q < p$, and in this case $c \approx B_1 + B_2 + \|S_u^*(\mathbf{1})\|_{q,w,(0,\infty)}/\|\mathbf{1}\|_{s,v,(0,\infty)}$, where*

$$B_1 := \left(\int_0^\infty \left(\int_x^\infty [\underline{u}]^q(t) w(t)\, dt \right)^{r/p} [\underline{u}]^q(x) V_*^{-r/p}(x) w(x)\, dx \right)^{1/r},$$

$$B_2 := \left(\int_0^\infty W^{r/p}(x) \left[\sup_{x\le\tau<\infty} \underline{u}(\tau) V_*^{-1/p}(\tau) \right]^r w(x)\, dx \right)^{1/r}.$$

Proof. It is easy to see that inequality (1.3) is satisfied with the best constant c if and only if

(3.5) $\|S_{\tilde{u}}f\|_{q,\tilde{w},(0,\infty)} \leq c\,\|f\|_{p,\tilde{v},(0,\infty)}, \ f \in \mathfrak{M}^{\downarrow}$

holds, where

$$\tilde{u}(t) = u\Big(\frac{1}{t}\Big), \quad \tilde{w}(t) = w\Big(\frac{1}{t}\Big)\frac{1}{t^2}, \quad \tilde{v}(t) = v\Big(\frac{1}{t}\Big)\frac{1}{t^2}, \ t > 0.$$

Using Theorem 3.3, and then applying substitution of variables mentioned above three times, we get the statement. $\square$

Theorem 3.5. *Let $0 < p, q < \infty$ and let $u \in \mathcal{W}(0,\infty) \cap C(0,\infty)$. Assume that $v, w \in \mathcal{W}(0,\infty)$ be such that $0 < V_*(x) < \infty$ and $0 < W_*(x) < \infty$ for all $x > 0$. Then (1.2) is satisfied with the best constant c if and only if:*

(i) *$p \leq q$, and in this case $c \approx A_1 + \|S_u(\mathbf{1})\|_{q,w,(0,\infty)}/\|\mathbf{1}\|_{p,v,(0,\infty)}$, where*

$$A_1 := \sup_{x>0} \Big(\Big[\sup_{0<\tau\leq x} \frac{u(\tau)^p}{V_*(\tau)^2} \Big]^{q/p} W_*(x) + \int_0^x \Big[\sup_{0<\tau\leq t} \frac{u(\tau)^p}{V_*(\tau)^2} \Big]^{q/p} w(t)\,dt \Big)^{1/q} [V_*]^{1/p}(x);$$

(ii) *$0 < q < p < \infty$, and in this case $c \approx A_2 + \|S_u(\mathbf{1})\|_{q,w,(0,\infty)}/\|\mathbf{1}\|_{p,v,(0,\infty)}$, where*

$$B_1 := \Big(\int_0^\infty \Big(\int_0^x \Big[\sup_{0<\tau\leq t} \frac{u(\tau)^p}{V_*(\tau)^2} \Big]^{q/p} w(t)\,dt \Big)^{r/p} [V_*]^{-r/p}(x) \Big[\sup_{0<\tau\leq x} \frac{u(\tau)^p}{V_*(\tau)^2} \Big]^{q/p} w(x)\,dx \Big)^{1/r},$$

$$B_2 := \Big(\int_0^\infty W_*^{r/p}(x) \Big[\sup_{0<\tau\leq x} \Big[\sup_{0<\tau\leq t} \frac{u(\tau)^p}{V_*(\tau)^2} \Big] V_*(\tau) \Big]^{r/p} w(x)\,dx \Big)^{1/r}.$$

Proof. It is easy to see that inequality (1.2) holds if and only if

(3.6) $\|S_{u^p}(f)\|_{q/p,w,(0,\infty)} \leq c^p \|f\|_{1,v,(0,\infty)}, \ f \in \mathfrak{M}^{\uparrow}$

holds. By Theorem 2.5 applied to the operator S_{u^p}, inequality (3.6) is satisfied with the best constant c if and only if both

$$\Big\| S_{u^p/V_*^2}\Big(\int_x^\infty h \Big) \Big\|_{q/p,w,(0,\infty)} \leq c\|h\|_{1,1/V_*,(0,\infty)}, \ h \in \mathfrak{M}^+,$$

and

$$\|S_u(\mathbf{1})\|_{q,w,(0,\infty)} \leq c\|\mathbf{1}\|_{p,v,(0,\infty)}$$

hold. It remains to apply Theorem 3.2. $\square$

The following "dual" statement holds true.

Theorem 3.6. *Let $0 < p, q < \infty$ and let $u \in \mathcal{W}(0,\infty) \cap C(0,\infty)$. Assume that $v, w \in \mathcal{W}(0,\infty)$ be such that $0 < V(x) < \infty$ and $0 < W(x) < \infty$ for all $x > 0$. Then (1.4) is satisfied with the best constant c if and only if:*

(i) *$p \leq q$, and in this case $c \approx A_1 + \|S_u^*(\mathbf{1})\|_{q,w,(0,\infty)}/\|\mathbf{1}\|_{p,v,(0,\infty)}$, where*

$$A_1 := \sup_{x>0} \Big(\Big[\sup_{x\leq\tau<\infty} \frac{u(\tau)^p}{V(\tau)^2} \Big]^{q/p} W(x) + \int_x^\infty \Big[\sup_{t\leq\tau<\infty} \frac{u(\tau)^p}{V(\tau)^2} \Big]^{q/p} w(t)\,dt \Big)^{1/q} V^{1/p}(x);$$

(ii) *$q < p$, and in this case $c \approx B_1 + B_2 + \|S_u^*(\mathbf{1})\|_{q,w,(0,\infty)}/\|\mathbf{1}\|_{p,v,(0,\infty)}$, where*

$$B_1 := \Big(\int_0^\infty \Big(\int_x^\infty \Big[\sup_{t\leq\tau<\infty} \frac{u(\tau)^p}{V(\tau)^2} \Big]^{q/p} w(t)\,dt \Big)^{r/p} V^{-r/p}(x) \Big[\sup_{x\leq\tau<\infty} \frac{u(\tau)^p}{V(\tau)^2} \Big]^{q/p} w(x)\,dx \Big)^{1/r},$$

$$B_2 := \Big(\int_0^\infty W^{r/p}(x) \Big[\sup_{x\leq\tau<\infty} \Big[\sup_{t\leq\tau<\infty} \frac{u(\tau)^p}{V(\tau)^2} \Big] V(\tau) \Big]^{r/p} w(x)\,dx \Big)^{1/r}.$$

Proof. Obviously, (1.4) is satisfied with the best constant c if and only if

$$\|S_{\tilde{u}}f\|_{q,\tilde{w},(0,\infty)} \le c \|f\|_{p,\tilde{v},(0,\infty)}, \ f \in \mathfrak{M}^{\uparrow}$$

holds, where

$$\tilde{u}(t) = u\left(\frac{1}{t}\right), \ \tilde{w}(t) = w\left(\frac{1}{t}\right)\frac{1}{t^2}, \ \tilde{v}(t) = v\left(\frac{1}{t}\right)\frac{1}{t^2}, \ t > 0.$$

Using Theorem 3.5, and then applying substitution of variables mentioned above three times, we get the statement. $\qquad\square$

4. ITERATED INEQUALITIES WITH SUPREMAL OPERATORS

In this section we characterize inequalities (1.5) and (1.7).

The following theorem is true.

Theorem 4.1. *Let* $1 < p < \infty$, $0 < q < \infty$ *and let* $u \in \mathcal{W}(0,\infty) \cap C(0,\infty)$. *Assume that* $v, w \in \mathcal{W}(0,\infty)$ *be such that*

$$0 < \int_0^x v^{1-p'}(t)\,dt < \infty \qquad and \qquad 0 < W_*(x) < \infty \qquad for\ all \qquad x > 0.$$

Recall that

$$\Phi[v; p](x) = \left(\int_0^x v^{1-p'}(t)\,dt\right)^{1/(p'+1)}, \ x > 0.$$

Denote by

$$\Phi_1(x) := \sup_{0<\tau\le x} u(\tau)\Phi^2[v; p](\tau), \ x > 0.$$

Then (1.5) *is satisfied with the best constant* c *if and only if:*

(i) $p \le q$, *and in this case* $c \approx A_1 + \|S_{u\Phi^2[v;p]}(\mathbf{1})\|_{q,w,(0,\infty)}/\|\mathbf{1}\|_{p,\phi[v;p],(0,\infty)}$, *where*

$$A_1 := \sup_{x>0}\left([\Phi_1]^q(x)W_*(x) + \int_0^x [\Phi_1]^q(t)w(t)\,dt\right)^{1/q}\Phi[v; p]^{-1/p}(x);$$

(ii) $q < p$, *and in this case* $c \approx B_1 + B_2 + \|S_{u\Phi^2[v;p]}(\mathbf{1})\|_{q,w,(0,\infty)}/\|\mathbf{1}\|_{p,\phi[v;p],(0,\infty)}$, *where*

$$B_1 := \left(\int_0^\infty \left(\int_0^x [\Phi_1]^q(t)w(t)\,dt\right)^{r/p}[\Phi_1]^q(x)\Phi[v; p]^{-r/p}(x)w(x)\,dx\right)^{1/r},$$

$$B_2 := \left(\int_0^\infty W_*^{r/p}(x)\left[\sup_{0<\tau\le x}\Phi_1(\tau)\Phi[v; p]^{-1/p}(\tau)\right]^r w(x)\,dx\right)^{1/r}.$$

Proof. By Theorem 2.6 applied to the operator S_u, inequality (1.5) with the best constant c holds if and only if the inequality

$$(4.1) \qquad \left\|S_{u\Phi^2[v;p]}(f)\right\|_{q,w,(0,\infty)} \le C \|f\|_{p,\phi[v;p],(0,\infty)}, \ f \in \mathfrak{M}^{\downarrow}$$

holds. Moreover, $c \approx C$. Now the statement follows by Theorem 3.3. $\qquad\square$

Theorem 4.2. *Let* $0 < q < \infty$ *and let* $u \in \mathcal{W}(0,\infty) \cap C(0,\infty)$. *Assume that* $v, w \in \mathcal{W}(0,\infty)$ *be such that* $0 < V(x) < \infty$ *and* $0 < W_*(x) < \infty$ *for all* $x > 0$. *Denote by*

$$V_1(x) := \sup_{0<\tau\le x} u(\tau)V^2(\tau), \ x > 0.$$

Then

$$(4.2) \qquad \left\|S_u\left(\int_0^x h\right)\right\|_{q,w,(0,\infty)} \le c \|h\|_{1,V^{-1},(0,\infty)},$$

is satisfied with the best constant c *if and only if:*

(i) $p \le q$, *and in this case* $c \approx A_1 + \|S_{uV^2}(\mathbf{1})\|_{q,w,(0,\infty)}/\|\mathbf{1}\|_{p,v,(0,\infty)}$, *where*

$$A_1 := \sup_{x>0}\left([V_1]^q(x)\int_x^\infty w(t)\,dt + \int_0^x [V_1]^q(t)w(t)\,dt\right)^{1/q}V^{-1/p}(x);$$

(ii) $q < p$, and in this case $c \approx B_1 + B_2 + \|S_{uV^2}(\mathbf{1})\|_{q,w,(0,\infty)}/\|\mathbf{1}\|_{p,v,(0,\infty)}$, where

$$B_1 := \left(\int_0^\infty \left(\int_0^x [V_1]^q(t)w(t)\,dt \right)^{r/p} [V_1]^q(x)V^{-r/p}(x)w(x)\,dx \right)^{1/r},$$

$$B_2 := \left(\int_0^\infty \left(\int_x^\infty w(t)\,dt \right)^{r/p} \left[\sup_{0<\tau\leq x} [V_1](\tau)V^{-1/p}(\tau) \right]^r w(x)\,dx \right)^{1/r}.$$

Proof. By Theorem 2.7 applied to the operator S_u, inequality (4.2) with the best constant c_{51} holds if and only if the inequality

(4.3) $$\left\| S_{uV^2}(f) \right\|_{q,w,(0,\infty)} \leq C \|f\|_{1,v,(0,\infty)}, \ f \in \mathfrak{M}^\downarrow$$

holds. Moreover, $c \approx C$. Now the statement follows by Theorem 3.3. $\qquad\square$

The following "dual" statements also hold true.

Theorem 4.3. *Let $1 < p < \infty$, $0 < q < \infty$ and let $u \in \mathcal{W}(0,\infty) \cap C(0,\infty)$. Assume that $v, w \in \mathcal{W}(0,\infty)$ be such that*

$$0 < \int_x^\infty v^{1-p'}(t)\,dt < \infty \qquad and \qquad 0 < W(x) < \infty \qquad for\ all \qquad x > 0.$$

Recall that

$$\Psi[v;p](x) = \left(\int_x^\infty v^{1-p'}(t)\,dt \right)^{1/(p'+1)}, \ x > 0.$$

Denote by

$$\Psi_1(x) := \sup_{x\leq\tau<\infty} u(\tau)\Psi^2[v;p](\tau), \ x > 0.$$

Then (1.7) is satisfied with the best constant c if and only if:
  (i) $p \leq q$, and in this case $c \approx A_1 + \|S_{u\Psi^2[v;p]}(\mathbf{1})\|_{q,w,(0,\infty)}/\|\mathbf{1}\|_{p,\psi[v;p],(0,\infty)}$, where

$$A_1 := \sup_{x>0} \left([\Psi_1]^q(x)W(x) + \int_x^\infty [\Psi_1]^q(t)w(t)\,dt \right)^{1/q} \Psi[v;p]^{-1/p}(x);$$

  (ii) $q < p$, and in this case $c \approx B_1 + B_2 + \|S_{u\Psi^2[v;p]}(\mathbf{1})\|_{q,w,(0,\infty)}/\|\mathbf{1}\|_{p,\psi[v;p],(0,\infty)}$, where

$$B_1 := \left(\int_0^\infty \left(\int_x^\infty [\Psi_1]^q(t)w(t)\,dt \right)^{r/p} [\Psi_1]^q(x)\Psi[v;p]^{-r/p}(x)w(x)\,dx \right)^{1/r},$$

$$B_2 := \left(\int_0^\infty W^{r/p}(x) \left[\sup_{x\leq\tau<\infty} \Psi_1(\tau)\Psi[v;p]^{-1/p}(\tau) \right]^r w(x)\,dx \right)^{1/r}.$$

Proof. Obviously, (1.7) is satisfied with the best constant c if and only if

$$\left\| S_{\tilde{u}}\left(\int_0^x h \right) \right\|_{q,\tilde{w},(0,\infty)} \leq c \|h\|_{p,\tilde{v},(0,\infty)}, \ h \in \mathfrak{M}^+$$

holds, where

$$\tilde{u}(t) = u\left(\frac{1}{t}\right), \ \tilde{w}(t) = w\left(\frac{1}{t}\right)\frac{1}{t^2}, \ \tilde{v}(t) = v\left(\frac{1}{t}\right)\frac{1}{t^2}, \ t > 0.$$

Using Theorem 4.1, and then applying substitution of variables mentioned above three times, we get the statement. $\qquad\square$

Theorem 4.4. *Let $0 < q < \infty$ and let $u \in \mathcal{W}(0,\infty) \cap C(0,\infty)$. Assume that $v, w \in \mathcal{W}(0,\infty)$ be such that $0 < V_*(x) < \infty$ and $0 < W_*(x) < \infty$ for all $x > 0$. Denote by*

$$V_1^*(x) := \sup_{x\leq\tau<\infty} u(\tau)V_*^2(\tau), \ x > 0.$$

Then

(4.4) $$\left\| S_u^*\left(\int_x^\infty h \right) \right\|_{q,w,(0,\infty)} \leq c \|h\|_{1,V_*^{-1},(0,\infty)},$$

is satisfied with the best constant c if and only if:

(i) $p \leq q$, *and in this case* $c \approx A_1 + \left\|S^*_{uV_*^2}(\mathbf{1})\right\|_{q,w,(0,\infty)}/\|\mathbf{1}\|_{p,v,(0,\infty)}$, *where*

$$A_1 := \sup_{x>0}\left([V_1^*]^q(x)W_*(x) + \int_0^x [V_1^*]^q(t)w(t)\,dt\right)^{1/q} V^{-1/p}(x);$$

(ii) $q < p$, *and in this case* $c \approx B_1 + B_2 + \left\|S^*_{uV_*^2}(\mathbf{1})\right\|_{q,w,(0,\infty)}/\|\mathbf{1}\|_{p,v,(0,\infty)}$, *where*

$$B_1 := \left(\int_0^\infty \left(\int_x^\infty [V_1^*]^q(t)w(t)\,dt\right)^{r/p} [V_1^*]^q(x)V_*^{-r/p}(x)w(x)\,dx\right)^{1/r},$$

$$B_2 := \left(\int_0^\infty W_*^{r/p}(x)\left[\sup_{x\leq\tau<\infty} [V_1^*](\tau)V^{-1/p}(\tau)\right]^r w(x)\,dx\right)^{1/r}.$$

Proof. By change of variables $x = 1/t$, it is easy to see that inequality (4.4) holds if and only if

(4.5) $$\left\|S_{\tilde{u}}\left(\int_0^x h\right)\right\|_{q,\tilde{w},(0,\infty)} \leq c\,\|h\|_{1,\tilde{V}^{-1},(0,\infty)}, \quad h \in \mathfrak{M}^+$$

holds, where

$$\tilde{u}(t) = u\left(\frac{1}{t}\right), \quad \tilde{w}(t) = w\left(\frac{1}{t}\right)\frac{1}{t^2}, \quad \tilde{V}(t) = \int_0^t v\left(\frac{1}{y}\right)\frac{1}{y^2}\,dy, \quad t > 0.$$

Applying Theorem 4.2, and then using substitution of variables mentioned above three times, we get the statement. $\square$

5. Hardy-operator involving suprema - $T_{u,b}$

In this section we give complete characterization of inequality (1.12).

5.1. The case $1 \leq p < \infty$. The following theorem is true.

Theorem 5.1. *Let* $0 < q \leq \infty$, $1 \leq p < \infty$ *and let* $u \in \mathcal{W}(0,\infty)\cap C(0,\infty)$. *Assume that* b, v, $w \in \mathcal{W}(0,\infty)$ *be such that*

$$0 < B(t) < \infty, \quad 0 < V(x) < \infty \text{ and } 0 < W(x) < \infty \text{ for all } x > 0.$$

Then inequality (1.12) *is satisfied with the best constant c if and only if:*

(i) $1 < p \leq q$, *and in this case* $c \approx A_1 + A_2$, *where*

$$A_1 := \sup_{x>0}\left(\left[\sup_{x\leq\tau<\infty}\frac{u(\tau)}{B(\tau)}\right]^q W(x) + \int_x^\infty \left[\sup_{t\leq\tau<\infty}\frac{u(\tau)}{B(\tau)}\right]^q w(t)\,dt\right)^{1/q}\left(\int_0^x\left(\frac{B(y)}{V(y)}\right)^{p'}v(y)\,dy\right)^{1/p'},$$

$$A_2 := \sup_{x>0}\left(\left[\sup_{x\leq\tau<\infty}\frac{u(\tau)}{V^2(\tau)}\right]^q W(x) + \int_x^\infty \left[\sup_{t\leq\tau<\infty}\frac{u(\tau)}{V^2(\tau)}\right]^q w(t)\,dt\right)^{1/q}\left(\int_0^x V^{p'}(y)v(y)\,dy\right)^{1/p'};$$

(ii) $1 = p \leq q$, *and in this case* $c \approx A_1 + A_2$, *where*

$$A_1 := \sup_{x>0}\left(\left[\sup_{x\leq\tau<\infty}\frac{u(\tau)}{B(\tau)}\right]^q W(x) + \int_x^\infty \left[\sup_{t\leq\tau<\infty}\frac{u(\tau)}{B(\tau)}\right]^q w(t)\,dt\right)^{1/q}\left(\sup_{0<y\leq x}\frac{B(y)}{V(y)}\right),$$

$$A_2 := \sup_{x>0}\left(\left[\sup_{x\leq\tau<\infty}\frac{u(\tau)}{V^2(\tau)}\right]^q W(x) + \int_x^\infty \left[\sup_{t\leq\tau<\infty}\frac{u(\tau)}{V^2(\tau)}\right]^q w(t)\,dt\right)^{1/q} V(x);$$

(iii) $1 < p$ *and* $q < p$, *and in this case* $c \approx B_1 + B_2 + B_3 + B_4$, *where*

$$B_1 := \left(\int_0^\infty\left(\int_x^\infty\left[\sup_{t\leq\tau<\infty}\frac{u(\tau)}{B(\tau)}\right]^q w(t)\,dt\right)^{r/p}\left[\sup_{x\leq\tau<\infty}\frac{u(\tau)}{B(\tau)}\right]^q\left(\int_0^x\left(\frac{B(y)}{V(y)}\right)^{p'}v(y)\,dy\right)^{r/p'}w(x)\,dx\right)^{1/r},$$

$$B_2 := \left(\int_0^\infty W^{r/p}(x)\left[\sup_{x\leq\tau<\infty}\left[\sup_{\tau\leq y<\infty}\frac{u(y)}{B(y)}\right]\left(\int_0^x\left(\frac{B(y)}{V(y)}\right)^{p'}v(y)\,dy\right)^{1/p'}\right]^r w(x)\,dx\right)^{1/r},$$

$$B_3 := \left(\int_0^\infty\left(\int_x^\infty\left[\sup_{t\leq\tau<\infty}\frac{u(\tau)}{V^2(\tau)}\right]^q w(t)\,dt\right)^{r/p}\left[\sup_{x\leq\tau<\infty}\frac{u(\tau)}{V^2(\tau)}\right]^q\left(\int_0^x V^{p'}(y)v(y)\,dy\right)^{r/p'}w(x)\,dx\right)^{1/r},$$

$$B_4 := \left(\int_0^\infty W^{r/p}(x) \left[\sup_{x \leq \tau < \infty} \left[\sup_{\tau \leq y < \infty} \frac{u(y)}{V^2(y)} \right] \left(\int_0^x V^{p'}(y)v(y)\,dy \right)^{1/p'} \right]^r w(x)\,dx \right)^{1/r}.$$

(iv) $q < 1 = p$, and in this case $c \approx B_1 + B_2 + B_3 + B_4$, where

$$B_1 := \left(\int_0^\infty \left(\int_x^\infty \left[\sup_{t \leq \tau < \infty} \frac{u(\tau)}{B(\tau)} \right]^q w(t)\,dt \right)^{r/p} \left[\sup_{x \leq \tau < \infty} \frac{u(\tau)}{B(\tau)} \right]^q \left(\sup_{0 < y \leq x} \frac{B(y)}{V(y)} \right)^r w(x)\,dx \right)^{1/r},$$

$$B_2 := \left(\int_0^\infty W^{r/p}(x) \left[\sup_{x \leq \tau < \infty} \left[\sup_{\tau \leq y < \infty} \frac{u(y)}{B(y)} \right] \left(\sup_{0 < y \leq x} \frac{B(y)}{V(y)} \right) \right]^r w(x)\,dx \right)^{1/r},$$

$$B_3 := \left(\int_0^\infty \left(\int_x^\infty \left[\sup_{t \leq \tau < \infty} \frac{u(\tau)}{V^2(\tau)} \right]^q w(t)\,dt \right)^{r/p} \left[\sup_{x \leq \tau < \infty} \frac{u(\tau)}{V^2(\tau)} \right]^q V^r(x)w(x)\,dx \right)^{1/r},$$

$$B_4 := \left(\int_0^\infty W^{r/p}(x) \left[\sup_{x \leq \tau < \infty} \left[\sup_{\tau \leq y < \infty} \frac{u(y)}{V^2(y)} \right] V(x) \right]^r w(x)\,dx \right)^{1/r}.$$

Proof. By Theorem 2.2, (1.12) holds iff both

$$(5.1) \qquad \left\| T_{u,b}\left(\int_x^\infty h \right) \right\|_{q,w,(0,\infty)} \leq c\|h\|_{p,V^p v^{1-p},(0,\infty)}, \qquad h \in \mathfrak{M}^+(0,\infty).$$

and

$$(5.2) \qquad \|T_{u,b}\mathbf{1}\|_{q,w,(0,\infty)} \leq c\|\mathbf{1}\|_{p,v,(0,\infty)}$$

hold.

Note that

$$T_{u,b}\left(\int_t^\infty h \right)(x) = \sup_{x \leq \tau < \infty} \frac{u(\tau)}{B(\tau)} \int_0^\tau \left(\int_s^\infty h(y)\,dy \right) b(s)\,ds$$

$$\approx \sup_{x \leq \tau < \infty} \frac{u(\tau)}{B(\tau)} \int_0^\tau h(y)B(y)\,dy + \sup_{x \leq \tau < \infty} u(\tau) \int_\tau^\infty h(s)\,ds$$

$$= S^*_{u/B}\left(\int_0^\tau hB \right) + S^*_u\left(\int_\tau^\infty h \right).$$

Hence, inequality (1.12) holds iff inequalities

$$(5.3) \qquad \left\| S^*_{u/B}\left(\int_0^\tau h \right) \right\|_{q,w,(0,\infty)} \leq c\|h\|_{p,B^{-p}V^p v^{1-p},(0,\infty)}, \qquad h \in \mathfrak{M}^+(0,\infty),$$

$$(5.4) \qquad \left\| S^*_u\left(\int_\tau^\infty h \right) \right\|_{q,w,(0,\infty)} \leq c\|h\|_{p,V^p v^{1-p},(0,\infty)}, \qquad h \in \mathfrak{M}^+(0,\infty).$$

and (5.2) hold.

Again by Theorem 2.2, (5.4) with (5.2) is equivalent to

$$(5.5) \qquad \|S^*_u f\|_{q,w,(0,\infty)} \leq c\|f\|_{p,v,(0,\infty)}, \qquad f \in \mathfrak{M}^\downarrow(0,\infty).$$

Now by Theorem 2.3, (5.5) is equivalent to

$$(5.6) \qquad \left\| S^*_{u/V^2}\left(\int_0^x h \right) \right\|_{q,w,(0,\infty)} \leq c\|h\|_{p,V^{-p}v^{1-p},(0,\infty)}, \qquad h \in \mathfrak{M}^+(0,\infty).$$

Consequently, (1.12) holds iff inequalities (5.3) and (5.6) hold.

(i) $p \leq q$. By Theorem 3.1, (5.3) and (5.6) hold if and only if both $A_1 < \infty$ and $A_2 < \infty$ hold, respectively.

(ii) $q < p$. By Theorem 3.1, (5.3) and (5.6) hold if and only if $B_i < \infty$, $i = 1, 2, 3, 4$ hold, respectively.

□

5.2. **The case** $0 < p < 1$. We start with a simple observation. If $0 < p \leq 1$ and $t \in (0, \infty)$, then

$$(5.7) \qquad \sup_{0<\tau\leq t} f(\tau)B(\tau) \leq \int_0^t f(y)b(y)\,dy \lesssim \left(\int_0^t f(y)^p B(y)^{p-1} b(y)\,dy \right)^{1/p}, \quad f \in \mathfrak{M}^\downarrow.$$

Since f is non-increasing, the first inequality in (5.7) is obvious. The second one follows, for instance, from the fact that (see, for instance, [5, Theorem 3.2], cf. also [22])

$$\sup_{f \in \mathfrak{M}^\downarrow : f \neq 0} \frac{\int_0^\infty f(x)g(x)\,dx}{\left(\int_0^\infty f(x)^p v(x)\,dx \right)^{1/p}} \approx \sup_{t>0} \left(\int_0^t g(x)\,dx \left(\int_0^t v(x)\,dx \right)^{-1/p} \right).$$

Our first aim is to prove a reduction theorem for the operator $T_{u,b}$. We first note that, using the monotonicity of $\int_0^t fb$ and interchanging the suprema, we get

$$(T_{u,b}g)(t) = \sup_{t\leq\tau<\infty} \frac{u(\tau)}{B(\tau)} \int_0^\tau g(y)b(y)\,dy = \sup_{t\leq\tau<\infty} \left(\sup_{\tau\leq x<\infty} \frac{u(x)}{B(x)} \right) \int_0^\tau g(y)b(y)\,dy, \qquad t \in (0, \infty).$$

As a consequence, we can safely assume that $u(x)/B(x)$ is non-increasing on $(0, \infty)$, since otherwise we would just replace $u(x)/B(x)$ by $\sup_{\tau\leq x<\infty} u(x)/B(x)$.

Theorem 5.2. *Let* $0 < p \leq 1$, $0 < q < \infty$. *Assume that* $u \in \mathcal{W}(0, \infty) \cap C(0, \infty)$ *and* b, v, $w \in \mathcal{W}(0, \infty)$ *be such that* $0 < B(t) < \infty$ *for all* $x > 0$. *Then the following three statements are equivalent:*

$$(5.8) \qquad \left(\int_0^\infty \left(\sup_{t\leq\tau<\infty} \frac{u(\tau)}{B(\tau)} \int_0^\tau f(y)b(y)\,dy \right)^q w(t)\,dt \right)^{1/q} \lesssim \left(\int_0^\infty f(\tau)^p v(\tau)\,d\tau \right)^{1/p}, \quad f \in \mathfrak{M}^\downarrow;$$

$$(5.9) \qquad \left(\int_0^\infty \left(\sup_{t\leq\tau<\infty} \left(\frac{u(\tau)}{B(\tau)} \right)^p \int_0^\tau f(y)B(y)^{p-1}b(y)\,dy \right)^{q/p} w(t)\,dt \right)^{1/q} \lesssim \left(\int_0^\infty f(\tau)v(\tau)\,d\tau \right)^{1/p}, \quad f \in \mathfrak{M}^\downarrow;$$

$$(5.10) \qquad \left(\int_0^\infty \left(\sup_{t\leq\tau<\infty} \frac{u(\tau)}{B(\tau)} \sup_{0<y\leq\tau} f(y)B(y) \right)^q w(t)\,dt \right)^{1/q} \lesssim \left(\int_0^\infty f(\tau)^p v(\tau)\,d\tau \right)^{1/p}, \quad f \in \mathfrak{M}^\downarrow.$$

Proof. Again, in view of (5.7), the implications (5.9) $\Rightarrow$ (5.8) $\Rightarrow$ (5.10) are obvious, and it just remains to show that (5.10) implies (5.9).

Suppose, thus, that (5.10) holds. Since $u(x)/B(x)$ is non-increasing, we have

$$\sup_{t\leq\tau<\infty} \frac{u(\tau)}{B(\tau)} \sup_{0<y\leq\tau} f(y)B(y) = \max \left\{ \sup_{t\leq\tau<\infty} \frac{u(\tau)}{B(\tau)} \sup_{0<y\leq t} f(y)B(y), \ \sup_{t\leq\tau<\infty} \frac{u(\tau)}{B(\tau)} \sup_{t<y\leq\tau} f(y)B(y) \right\}$$

$$= \max \left\{ \frac{u(t)}{B(t)} \sup_{0<y\leq t} f(y)B(y), \ \sup_{t\leq y<\infty} f(y)B(y) \sup_{y\leq\tau<\infty} \frac{u(\tau)}{B(\tau)} \right\}$$

$$= \max \left\{ \frac{u(t)}{B(t)} \sup_{0<y\leq t} f(y)B(y), \ \sup_{t\leq y<\infty} f(y)u(y) \right\}.$$

Hence, (5.10) breaks down into the following two inequalities:

$$(5.11) \qquad \left(\int_0^\infty \left(\sup_{0<y\leq t} f(y)B(y) \right)^q w(t) \left(\frac{u(t)}{B(t)} \right)^q dt \right)^{1/q} \lesssim \left(\int_0^\infty f(\tau)^p v(\tau)\,d\tau \right)^{1/p}, \quad f \in \mathfrak{M}^\downarrow,$$

$$(5.12) \qquad \left(\int_0^\infty \left(\sup_{t\leq y<\infty} f(y)u(y) \right)^q w(t)\,dt \right)^{1/q} \lesssim \left(\int_0^\infty f(\tau)^p v(\tau)\,d\tau \right)^{1/p}, \quad f \in \mathfrak{M}^\downarrow.$$

Obviously, (5.11) and (5.12) are equivalent to

$$(5.13) \qquad \left(\int_0^\infty \left(\sup_{0<y\leq t} f(y)B(y)^p \right)^{q/p} w(t) \left(\frac{u(t)}{B(t)} \right)^q dt \right)^{p/q} \lesssim \int_0^\infty f(\tau)v(\tau)\,d\tau, \ f \in \mathfrak{M}^\downarrow,$$

$$(5.14) \qquad \left(\int_0^\infty \left(\sup_{t\leq y<\infty} f(y)u(y)^p \right)^{q/p} w(t)\,dt \right)^{p/q} \lesssim \int_0^\infty f(\tau)v(\tau)\,d\tau, \ f \in \mathfrak{M}^\downarrow.$$

(i) Let $p \leq q$. By Theorem 3.3, (5.13) holds iff both

$$(5.15) \qquad \sup_{x>0}\left(\int_0^x u(t)^q w(t)\,dt + B(x)^q \int_x^\infty \left[\frac{u(t)}{B(t)}\right]^q w(t)\,dt\right)^{1/q} V^{-1/p}(x) < \infty$$

and

$$(5.16) \qquad \left(\int_0^\infty u(t)^q w(t)\,dt\right)^{1/q} \lesssim \left(\int_0^\infty v(\tau)\,d\tau\right)^{1/p}$$

hold.

By Theorem 3.6, (5.14) holds iff both

$$(5.17) \qquad \sup_{x>0}\left(\left[\sup_{x\leq y<\infty}\frac{u(y)^p}{V^2(y)}\right]^{q/p}\int_0^x w(t)\,dt + \int_x^\infty\left[\sup_{t\leq y<\infty}\frac{u(y)^p}{V^2(y)}\right]^{q/p}w(t)\,dt\right)^{1/q}V^{1/p}(x) < \infty$$

and

$$(5.18) \qquad \left(\int_0^\infty \left(\sup_{t\leq\tau<\infty}u(\tau)^p\right)^{q/p} w(t)\,dt\right)^{1/q} \lesssim \left(\int_0^\infty v(\tau)\,d\tau\right)^{1/p}$$

hold.

On the other hand, by Theorem 5.1, (5.9) holds iff inequalities

$$(5.19) \qquad \sup_{x>0}\left(\left[\frac{u(x)}{B(x)}\right]^q\int_0^x w(t)\,dt + \int_x^\infty\left[\frac{u(t)}{B(t)}\right]^q w(t)\,dt\right)^{1/q}\sup_{0<\tau\leq x}\frac{B(\tau)}{V^{1/p}(\tau)} < \infty$$

$$(5.20) \qquad \sup_{x>0}\left(\left[\sup_{x\leq\tau<\infty}\frac{u(\tau)^p}{V^2(\tau)}\right]^{q/p}\int_0^x w(t)\,dt + \int_x^\infty\left[\sup_{t\leq\tau<\infty}\frac{u(\tau)^p}{V^2(\tau)}\right]^{q/p}w(t)\,dt\right)^{1/q}V^{1/p}(x) < \infty$$

hold.

We will thus be done if we can show that (5.15) together with (5.17) imply (5.19). The latter can be proved as follows:

Since

$$\sup_{x>0}\left(\int_x^\infty\left[\frac{u(t)}{B(t)}\right]^q w(t)\,dt\right)^{1/q}\sup_{0<\tau\leq x}\frac{B(\tau)}{V^{1/p}(\tau)} = \sup_{x>0}\left(\int_x^\infty\left[\frac{u(t)}{B(t)}\right]^q w(t)\,dt\right)^{1/q}\frac{B(x)}{V^{1/p}(x)},$$

it remains to show that

$$\sup_{x>0}\frac{u(x)}{B(x)}\left(\int_0^x w(t)\,dt\right)^{1/q}\sup_{0<\tau\leq x}\frac{B(\tau)}{V^{1/p}(\tau)}$$

$$\lesssim \sup_{x>0}\frac{u(x)}{V^{1/p}(x)}\left(\int_0^x w(t)\,dt\right)^{1/q} + \sup_{x>0}\frac{B(x)}{V^{1/p}(x)}\left(\int_x^\infty\left[\frac{u(t)}{B(t)}\right]^q w(t)\,dt\right)^{1/q}.$$

Interchanging the suprema, using the monotonicity of u/B, we get that

$$\sup_{x>0}\frac{u(x)}{B(x)}\left(\int_0^x w(t)\,dt\right)^{1/q}\sup_{0<\tau\leq x}\frac{B(\tau)}{V^{1/p}(\tau)}$$

$$= \sup_{\tau>0}\frac{B(\tau)}{V^{1/p}(\tau)}\sup_{\tau\leq x<\infty}\frac{u(x)}{B(x)}\left(\int_0^x w(t)\,dt\right)^{1/q}$$

$$\lesssim \sup_{\tau>0}\frac{B(\tau)}{V^{1/p}(\tau)}\left(\sup_{\tau\leq x<\infty}\frac{u(x)}{B(x)}\right)\left(\int_0^\tau w(t)\,dt\right)^{1/q} + \sup_{\tau>0}\frac{B(\tau)}{V^{1/p}(\tau)}\sup_{\tau\leq x<\infty}\frac{u(x)}{B(x)}\left(\int_\tau^x w(t)\,dt\right)^{1/q}$$

$$\lesssim \sup_{\tau>0}\frac{u(\tau)}{V^{1/p}(\tau)}\left(\int_0^\tau w(t)\,dt\right)^{1/q} + \sup_{\tau>0}\frac{B(\tau)}{V^{1/p}(\tau)}\sup_{\tau\leq x<\infty}\left(\int_\tau^x\left[\frac{u(t)}{B(t)}\right]^q w(t)\,dt\right)^{1/q}$$

$$= \sup_{\tau>0}\frac{u(\tau)}{V^{1/p}(\tau)}\left(\int_0^\tau w(t)\,dt\right)^{1/q} + \sup_{\tau>0}\frac{B(\tau)}{V^{1/p}(\tau)}\left(\int_\tau^\infty\left[\frac{u(t)}{B(t)}\right]^q w(t)\,dt\right)^{1/q}.$$

(ii) Let $q < p$. By Theorem 3.3, (5.13) holds iff both

$$\int_0^\infty \left(\int_0^x u(t)^q w(t)\, dt \right)^{r/p} u(x)^q V^{-r/p}(x) w(x)\, dx < \infty,$$

$$\int_0^\infty \left(\int_x^\infty \left(\frac{u(t)}{B(t)} \right)^q w(t)\, dt \right)^{r/p} \left[\sup_{0<\tau\le x} \frac{B(\tau)}{V^{1/p}(\tau)} \right]^r w(x) \left(\frac{u(x)}{B(x)} \right)^q dx < \infty.$$

By Theorem 3.6, (5.14) holds iff inequalities

$$\int_0^\infty \left(\int_x^\infty \left[\sup_{t\le y<\infty} \frac{u(y)^p}{V^2(y)} \right]^{q/p} w(t)\, dt \right)^{r/p} \left[\sup_{x\le y<\infty} \frac{u(y)^p}{V^2(y)} \right]^{q/p} V^{r/p}(x) w(x)\, dx < \infty,$$

$$\int_0^\infty \left(\int_0^x w(t)\, dt \right)^{r/p} \left(\sup_{x\le\tau<\infty} \left[\sup_{\tau\le y<\infty} \frac{u(y)^p}{V^2(y)} \right] V(\tau) \right)^{r/p} w(x)\, dx < \infty.$$

On the other hand, by Theorem 5.1, (5.9) holds iff

$$\int_0^\infty \left(\int_x^\infty \left[\frac{u(t)}{B(t)} \right]^q w(t)\, dt \right)^{r/p} \left[\sup_{0<t\le\tau} \frac{B(t)^p}{V(t)} \right]^{r/p} w(x) \left[\frac{u(x)}{B(x)} \right]^q dx < \infty,$$

$$\int_0^\infty \left(\int_0^x w(t)\, dt \right)^{r/p} \left(\sup_{x\le\tau<\infty} \left[\frac{u(\tau)}{B(\tau)} \right]^p \left[\sup_{0<t\le\tau} \frac{B(t)^p}{V(t)} \right] \right)^{r/p} w(x)\, dx < \infty,$$

$$\int_0^\infty \left(\int_x^\infty \left[\sup_{t\le y<\infty} \frac{u(y)^p}{V^2(y)} \right]^{q/p} w(t)\, dt \right)^{r/p} \left[\sup_{x\le y<\infty} \frac{u(y)^p}{V^2(y)} \right]^{q/p} V^{r/p}(x) w(x)\, dx < \infty,$$

$$\int_0^\infty \left(\int_0^x w(t)\, dt \right)^{r/p} \left(\sup_{x\le\tau<\infty} \left[\sup_{\tau\le y<\infty} \frac{u(y)^p}{V^2(y)} \right] V(\tau) \right)^{r/p} w(x)\, dx$$

$$= \int_0^\infty \left(\int_0^x w(t)\, dt \right)^{r/p} \left(\sup_{x\le\tau<\infty} \frac{u(\tau)^p}{V(\tau)} \right)^{r/p} w(x)\, dx < \infty.$$

Obviously, it remains to show that

$$\int_0^\infty \left(\int_0^x w(t)\, dt \right)^{r/p} \left(\sup_{x\le\tau<\infty} \left[\frac{u(\tau)}{B(\tau)} \right]^p \left[\sup_{0<t\le\tau} \frac{B(t)^p}{V(t)} \right] \right)^{r/p} w(x)\, dx$$

$$\lesssim \int_0^\infty \left(\int_x^\infty \left(\frac{u(t)}{B(t)} \right)^q w(t)\, dt \right)^{r/p} \left[\sup_{0<\tau\le x} \frac{B(\tau)}{V^{1/p}(\tau)} \right]^r w(x) \left(\frac{u(x)}{B(x)} \right)^q dx$$

$$+ \int_0^\infty \left(\int_0^x w(t)\, dt \right)^{r/p} \left(\sup_{x\le\tau<\infty} \frac{u(\tau)^p}{V(\tau)} \right)^{r/p} w(x)\, dx.$$

We will prove the assertion only in the case when $\int_0^\infty w(\tau)\, d\tau = \infty$. Let $\{x_k\}$ be such that $\int_0^{x_k} w(\tau)\, d\tau = 2^k$.
Then

$$\int_0^\infty \left(\int_0^x w(t)\, dt \right)^{r/p} \left(\sup_{x\le\tau<\infty} \left[\frac{u(\tau)}{B(\tau)} \right]^p \left[\sup_{0<t\le\tau} \frac{B(t)^p}{V(t)} \right] \right)^{r/p} w(x)\, dx$$

$$\approx \sum_{k\in\mathbb{Z}} 2^{kr/q} \left(\sup_{x_k\le\tau<\infty} \left[\frac{u(\tau)}{B(\tau)} \right]^p \left[\sup_{0<t\le\tau} \frac{B(t)^p}{V(t)} \right] \right)^{r/p}.$$

Note that

$$\sup_{x_k\le\tau<\infty} \left[\frac{u(\tau)}{B(\tau)} \right]^p \left[\sup_{0<t\le\tau} \frac{B(t)^p}{V(t)} \right] \approx \sup_{x_k\le\tau<\infty} \left[\frac{u(\tau)}{B(\tau)} \right]^p \left[\sup_{0<t\le x_k} \frac{B(t)^p}{V(t)} \right] + \sup_{x_k\le\tau<\infty} \left[\frac{u(\tau)}{B(\tau)} \right]^p \left[\sup_{x_k\le t\le\tau} \frac{B(t)^p}{V(t)} \right]$$

$$= \left[\frac{u(x_k)}{B(x_k)} \right]^p \left[\sup_{0<t\le x_k} \frac{B(t)^p}{V(t)} \right] + \sup_{x_k\le t<\infty} \frac{B(t)^p}{V(t)} \sup_{t\le\tau<\infty} \left[\frac{u(\tau)}{B(\tau)} \right]^p$$

$$= \left[\frac{u(x_k)}{B(x_k)} \right]^p \left[\sup_{0<t\le x_k} \frac{B(t)^p}{V(t)} \right] + \sup_{x_k\le t<\infty} \frac{u(t)^p}{V(t)}$$

$$\approx \left[\frac{u(x_k)}{B(x_k)}\right]^p \left[\sup_{0<t\le x_{k-1}} \frac{B(t)^p}{V(t)}\right] + \left[\frac{u(x_k)}{B(x_k)}\right]^p \left[\sup_{x_{k-1}<t\le x_k} \frac{B(t)^p}{V(t)}\right] + \sup_{x_k\le t<\infty} \frac{u(t)^p}{V(t)}$$

$$\le \left[\frac{u(x_k)}{B(x_k)}\right]^p \left[\sup_{0<t\le x_{k-1}} \frac{B(t)^p}{V(t)}\right] + \sup_{x_{k-1}\le t<x_k} \frac{u(t)^p}{V(t)} + \sup_{x_k\le t<\infty} \frac{u(t)^p}{V(t)}$$

$$\le \left[\frac{u(x_k)}{B(x_k)}\right]^p \left[\sup_{0<t\le x_{k-1}} \frac{B(t)^p}{V(t)}\right] + \sup_{x_{k-1}\le t<\infty} \frac{u(t)^p}{V(t)}.$$

Hence,

$$\int_0^\infty \left(\int_0^x w(t)\,dt\right)^{r/p} \left(\sup_{x\le\tau<\infty}\left[\frac{u(\tau)}{B(\tau)}\right]^p \left[\sup_{0<t\le\tau}\frac{B(t)^p}{V(t)}\right]\right)^{r/p} w(x)\,dx$$

$$\lesssim \sum_{k\in\mathbb{Z}} 2^{kr/q}\left(\left[\frac{u(x_k)}{B(x_k)}\right]^p\left[\sup_{0<t\le x_{k-1}}\frac{B(t)^p}{V(t)}\right]\right)^{r/p} + \sum_{k\in\mathbb{Z}} 2^{kr/q}\left(\sup_{x_{k-1}\le t<\infty}\frac{u(t)^p}{V(t)}\right)^{r/p}$$

$$\approx \sum_{k\in\mathbb{Z}}\left(\int_{x_{k-1}}^{x_k}\left(\int_x^{x_k} w\right)^{r/p} w(x)\,dx\right)\left(\left[\frac{u(x_k)}{B(x_k)}\right]^p\left[\sup_{0<t\le x_{k-1}}\frac{B(t)^p}{V(t)}\right]\right)^{r/p}$$

$$+ \sum_{k\in\mathbb{Z}}\left(\int_{x_{k-2}}^{x_{k-1}}\left(\int_{x_k}^x w\right)^{r/p} w(x)\,dx\right)\left(\sup_{x_{k-1}\le t<\infty}\frac{u(t)^p}{V(t)}\right)^{r/p}$$

$$\lesssim \sum_{k\in\mathbb{Z}}\int_{x_{k-1}}^{x_k}\left(\int_x^\infty\left(\frac{u(t)}{B(t)}\right)^q w(t)\,dt\right)^{r/p}\left[\sup_{0<\tau\le x}\frac{B(\tau)}{V^{1/p}(\tau)}\right]^r\left[\frac{u(x)}{B(x)}\right]^q w(x)\,dx$$

$$+ \sum_{k\in\mathbb{Z}}\int_{x_{k-1}}^{x_{k+1}}\left(\int_0^x w(t)\,dt\right)^{r/p}\left(\sup_{x\le\tau<\infty}\frac{u(\tau)^p}{V(\tau)}\right)^{r/p} w(x)\,dx$$

$$\lesssim \int_0^\infty\left(\int_x^\infty\left(\frac{u(t)}{B(t)}\right)^q w(t)\,dt\right)^{r/p}\left[\sup_{0<\tau\le x}\frac{B(\tau)}{V^{1/p}(\tau)}\right]^r w(x)\left(\frac{u(x)}{B(x)}\right)^q dx$$

$$+ \int_0^\infty\left(\int_0^x w(t)\,dt\right)^{r/p}\left(\sup_{x\le\tau<\infty}\frac{u(\tau)^p}{V(\tau)}\right)^{r/p} w(x)\,dx.$$

$\square$

Remark 5.3. Note that Theorem 5.2, namely the fact that (5.9) $\Leftrightarrow$ (5.8) $\Leftrightarrow$ (5.10), when $b \equiv 1$, was proved in [15].

As a corollary we obtain that for all the three operators mentioned in (5.7), the corresponding weighted inequalities are equivalent. It is worth noticing that this is not so when $p > 1$.

Corollary 5.4. *Assume that* $0 < p \le 1$, $0 < q < \infty$, *and* $v, w \in \mathcal{W}(0,\infty)$. *Let* b *be a weight on* $(0,\infty)$ *such that* $0 < B(t) < \infty$ *for every* $t \in (0,\infty)$. *Then the following three statements are equivalent:*

(5.21) $$\left(\int_0^\infty\left(\int_0^t f(\tau)b(\tau)\,d\tau\right)^q w(t)\,dt\right)^{1/q} \lesssim \left(\int_0^\infty f(\tau)^p v(\tau)\,d\tau\right)^{1/p}, \quad f\in\mathfrak{M}^\downarrow;$$

(5.22) $$\left(\int_0^\infty\left(\int_0^t f(\tau)^p B(\tau)^{p-1}b(\tau)\,d\tau\right)^{q/p} w(t)\,dt\right)^{1/q} \lesssim \left(\int_0^\infty f(\tau)^p v(\tau)\,d\tau\right)^{1/p}, \quad f\in\mathfrak{M}^\downarrow;$$

(5.23) $$\left(\int_0^\infty\left(\sup_{0<\tau\le t} f(\tau)B(\tau)\right)^q w(t)\,dt\right)^{1/q} \lesssim \left(\int_0^\infty f(\tau)^p v(\tau)\,d\tau\right)^{1/p}, \quad f\in\mathfrak{M}^\downarrow.$$

This fact was proved in [15, Theorem 2.1], when $b \equiv 1$. Recently, in [16, Theorem 3.9], it was proved that (5.21) $\Leftrightarrow$ (5.23) for more general Volterra operators with continuous Oinarov kernels in the case when $0 < q < p \le 1$.

Proof. The proof immediately follows from Theorem 5.2 taking $u \equiv 1$. $\square$

By the way we have proved the following statement.

Theorem 5.5. *Let* $0 < p \le 1$, $0 < q < \infty$. *Assume that* $u \in \mathcal{W}(0, \infty) \cap C(0, \infty)$ *and* b, v, $w \in \mathcal{W}(0, \infty)$ *be such that* $0 < V(t) < \infty$ *and* $0 < B(t) < \infty$ *for all* $x > 0$. *Then inequality* (1.12) *is satisfied with the best constant* c *if and only if:*

(i) $p \le q$, *and in this case* $c \approx A_1 + A_2$, *where*

$$A_1 := \sup_{x>0}\left(\left[\sup_{x \le \tau < \infty}\frac{u(\tau)}{B(\tau)}\right]^q \int_0^x w(t)\,dt + \int_x^\infty\left[\sup_{t \le \tau < \infty}\frac{u(\tau)}{B(\tau)}\right]^q w(t)\,dt\right)^{1/q} \sup_{0 < y \le x}\frac{B(y)}{V^{1/p}(y)};$$

$$A_2 := \sup_{x>0}\left(\left[\sup_{x \le y < \infty}\frac{u(y)^p}{V^2(y)}\right]^{q/p}\int_0^x w(t)\,dt + \int_x^\infty\left[\sup_{t \le y < \infty}\frac{u(y)^p}{V^2(y)}\right]^{q/p} w(t)\,dt\right)^{1/q} V^{1/p}(x);$$

(ii) $q < p$, *and in this case* $c \approx B_1 + B_2 + B_3 + B_4$, *where*

$$B_1 := \left(\int_0^\infty\left(\int_0^x w(t)\,dt\right)^{r/p}\left[\sup_{x \le \tau < \infty}\left[\sup_{\tau \le y < \infty}\frac{u(y)}{B(y)}\right]^q\left(\sup_{0 < y \le x}\frac{B(y)}{V^{1/p}(y)}\right)\right]^r w(x)\,dx\right)^{1/r},$$

$$B_2 := \left(\int_0^\infty\left(\int_x^\infty\left[\sup_{t \le \tau < \infty}\frac{u(\tau)}{B(\tau)}\right]^q w(t)\,dt\right)^{r/p}\left[\sup_{0 < \tau \le x}\frac{B(\tau)}{V^{1/p}(\tau)}\right]^r\left[\sup_{x \le \tau < \infty}\frac{u(\tau)}{B(\tau)}\right]^q w(x)\,dx\right)^{1/r},$$

$$B_3 := \left(\int_0^\infty\left(\int_0^x w(t)\,dt\right)^{r/p}\left(\sup_{x \le \tau < \infty}\left[\sup_{\tau \le y < \infty}\frac{u(y)^p}{V^2(y)}\right]V(\tau)\right)^{r/p} w(x)\,dx\right)^{1/r},$$

$$B_4 := \left(\int_0^\infty\left(\int_x^\infty\left[\sup_{t \le y < \infty}\frac{u(y)^p}{V^2(y)}\right]^{q/p} w(t)\,dt\right)^{r/p}\left[\sup_{x \le y < \infty}\frac{u(y)^p}{V^2(y)}\right]^{q/p} V^{r/p}(x)w(x)\,dx\right)^{1/r}.$$

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INSTITUTE OF MATHEMATICS, ACADEMY OF SCIENCES OF THE CZECH REPUBLIC, ŽITNÁ 25, 115 67 PRAHA 1, CZECH REPUBLIC
E-mail address: gogatish@math.cas.cz

DEPARTMENT OF MATHEMATICS, FACULTY OF SCIENCE AND ARTS, KIRIKKALE UNIVERSITY, 71450 YAHSIHAN, KIRIKKALE, TURKEY
E-mail address: rzamustafayev@gmail.com